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# Feedback Control of an Overdetermined Storage System

**DOUGLASS J. WILDE**
**Stanford University, Stanford, California**

**Although it is already known how to apply feedforward control to an overdetermined system (2), feedback control is more difficult because overdeterminancy involves more measured variables than controllers. Feedback control is possible for a system of production units separated by storage tanks holding intermediate products and perturbed by random external upsets (1). Two types of feedback control are studied: local control in which each manipulated variable is controlled by the levels in the two tanks immediately adjacent, and central control in which all the levels simultaneously affect each manipulated variable. The behavior of these systems for three-action linear controllers is described. Proportional local control is shown to have a structure similar to those of the unrefluxed countercurrent separation processes of the chemical industry. Such systems are therefore inherently stable. Central control, although requiring a more complicated equipment arrangement, actually has a simpler dynamic behavior because it cancels out interactions between controllers.**

An overdetermined control system is one in which the number of variables to be controlled, that is held within predetermined limits, exceeds the number of variables manipulated by the controllers (2). Past work on overdetermined systems has been confined to feedforward schemes in which the manipulated variables are adjusted as functions of the uncontrollable disturbances upsetting the system (*1, 2, 3*). This article shows how to control an overdetermined system by feedback, that is by making the manipulated variables depend on the variables to be regulated. Since in an overdetermined system there are more measurements of controlled variables than there are controllers, how to apply feedback control is not obvious.

The particular overdetermined system considered is a set of production units in series separated from each other by storage tanks holding intermediate product (*1*). Any or all of the levels in these tanks are subjected to random disturbances from outside the system. The control problem is to regulate the production rates to counteract the disturbances and prevent any tank from becoming completely full or empty. This problem, quite common in the chemical industry, is usually handled now on an ad hoc basis by planning and coordinating groups.

After a brief review of previous work, definition of terms, and mathematical statement of the control problem, two feedback schemes are analyzed. The first, local control, makes each production rate depend only on the levels in the tanks holding feed and product for that particular production unit. The second is called *central control* because each production rate is influenced by the levels of all the tanks in the system. Local control is simpler to implement but more difficult to analyze. Despite the complicated interactions between the manipulated flow rates resulting from local control, it is possible to show that local control is inherently stable if linear controllers are used. On the other hand, the more intricate physical layout required by central control is offset by the surprising simplicity of its dynamic behavior, for the extra information channels actually cancel out all controller interactions. Thus while local control has the slowness of a high-order system, central control behaves more like a set of independent (and inherently stable) first-order systems. Incidentally, local proportional control gives a system with a structure, that is signal flow diagram, resembling that of unreflexed countercurrent multistage separation processes, for which the dynamic behavior is already known in some detail (*7, 8*).

Full proofs are outlined only for the major results involving the asymptotic behavior of local control and the absence of controller interaction in centrally controlled systems. Consequences that would be fairly obvious to engineers experienced in linear control theory are merely

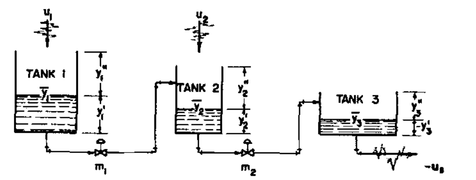


**Fig. 1. Flow diagram for three-tank inventory system.**

stated and justified informally. This permits brevity without sacrificing completeness when discussing, for example, the roles of conventional integral and derivative control actions in counteracting offset and saturation in a practical situation.

With the theory fully developed, the article ends by discussing possible applications to automatic inventory control systems. The reader may wish to study this concluding section before tackling the mathematical proofs in the body of the paper.

## AN OVERDETERMINED STORAGE SYSTEM

Consider a set of $n$ storage tanks during a time interval of duration $T$, at the beginning of which the amount of material in the $i^{th}$ tank $(i = 1, 2, \ldots, n)$ is $\overline{y}_i$. It is predicted that material will flow into tank $i$ at an average rate $\overline{u}_i$ during the time interval, flow out of any tank being indicated by a negative rate. The $\overline{u}_i$ are known collectively as the *forecast*. Additional material can also enter each tank at an instantaneous flow rate $u_i$ which, fluctuating unpredictably and uncontrollably from one instant to the next, will be called the *uncontrollable variable* (or *disturbance*) for tank $i$. These upsetting influences might be caused by forecasting errors or unpredictable changes in sales and procurement plans.

Suppose that material from tank $i$ is to be processed, the product being deposited in tank $i + 1$ $(i = 1, \ldots, n - 1)$. Let $\overline{m}_i$ $(i = 1, \ldots, n - 1)$ be the predicted average rate at which tank $i$ is to be depleted by this processing or transfer during the time interval, and let $k_i$ be the number of units of material in tank $i + 1$ generated by the processing of a unit of material from $i$. For simplicity of exposition it can be assumed without loss of generality that nonnegative $\overline{m}_i$ (known collectively as the *production schedule*) can be chosen to compensate exactly for the forecasted demands $\overline{u}_i$ on every tank. That is

$$\overline{u}_i + k_{i-1}\,\overline{m}_{i-1} - \overline{m}_i = 0; \quad i = 1, \ldots, n \tag{1}$$

with the convention that

$$\overline{m}_0 = \overline{m}_n \equiv 0 \tag{2}$$

The instantaneous inventory deviation $y_i$, the difference between the actual amount in tank $i$ and the original amount $\overline{y}_i$, can be expressed independently both of the forecasted $\overline{u}_i$ and the scheduled $\overline{m}_i$. Let $m_i$, known as the *instantaneous production deviation*, be the difference between the actual flow rate and the scheduled rate $\overline{m}_i$. Then a material balance around tank $i$, combined with Equation (1), gives $y_i$ at time $t$ as

$$y_i = \int_0^t (u_i + k_{i-1}\,m_{i-1} = m_i)\,dt; \quad i = 1, \ldots, n \tag{3}$$

where

$$m_0 = m_n \equiv 0 \text{ for all } 0 \leqq t \leqq T \tag{4}$$

All terms of the integrand are functions of the dummy time variable $t'$, and $y_i$ depends on the elapsed time $t$.

A control problem arises when the inventory deviations must be held within certain predetermined limits throughout the interval. This corresponds to avoiding having either too much or too little in any tank at any time. These restrictions may be expressed as

$$y_i' \leqq y_i \leqq y_i'' \text{ for all } i = 1, \ldots, n \text{ and all } 0 \leqq t \leqq T \tag{5}$$

where $y_i'$ is the lower and $y_i''$ the upper limit on the deviation $y_i$. When all of the inequalities (5) hold, the system is said to be operating satisfactorily. On the other hand, unsatisfactory operation occurs whenever any inequality is violated. Figure 1 is a flow diagram of a three tank overdetermined system.

The upsetting influence of the randomly fluctuating uncontrolled variables $u_i$ is to be counteracted as much as possible by manipulating the adjustable production variables $m_i$. Suppose one measures the effectiveness of any control scheme by the probability of satisfactory operation associated with it. Then any optimal scheme will be such that no inventory $y_i$ reaches a limit before any of the others (*1*). Such performance can be obtained by feedforward control in which the manipulated $m_i$ are made piecewise linear functions of the measured disturbances $u_i$ (*1*). A simpler linear (rather than piecewise linear) control law can be used with nearly optimal performance if it can be assumed that for all tanks $(i = 1, \ldots, n)$ either

$$-y_i' \geqq y_i'' > 0 \tag{6a}$$

or

$$y_i'' \geqq -y_i' > 0 \tag{6b}$$

Since one or the other of these conditions can always be attained by redistributing material between the tanks, only the simpler linear law will be considered here.

## ASYMPTOTIC CONTROL CONSTANTS

No generality is lost in taking Equation (6*a*) to be the case rather than Equation (6*b*) and setting $y_i''$ equal to unity. Then the linear control law corresponding to Equation (6*a*) is

$$m_i = \sum_{j=1}^{n} a_{ij} u_j; \quad i = 1, \ldots, n - 1 \tag{7}$$

where the asymptotic control constants $a_{ij}$ are calculated from the constants $k_i$. It is typographically convenient to express the $a_{ij}$ in terms of a two-parameter function $S(u, v)$ called *the sum of products from u to v*, defined by

$$S(u, v) = \sum_{g=u}^{v} \prod_{h=g}^{v-1} k_h = k_u\,k_{u+1} \ldots k_{v-1} + k_{u+1} \ldots k_{v-1} + \ldots + 1, \; u = 1, \ldots, n, \; u \leqq v \tag{8}$$

In terms of these sum of products functions the control constants may be written as

$$a_{ij} \equiv \begin{cases} S(i+1, n) \prod_{f=j}^{i-1} k_f \Big/ S(1, n) \text{ for } n \geqq i \geqq j \geqq 1 & (9a) \\ -S(1, i) \prod_{f=j}^{n-1} k_f \Big/ S(1, n) \text{ for } 1 \leqq i < j \leqq n & (9b) \end{cases}$$

where the convention has been adopted that

$$\prod_{f=j}^{j-1} k_f \equiv 1; \quad j = 1, \ldots, n \tag{10}$$

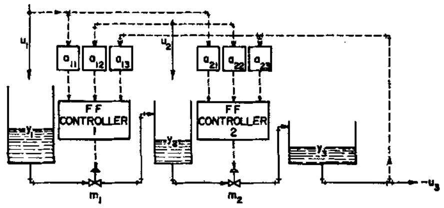


**Fig. 2. Flow and instrument diagram for feedforward control.**

For conservative systems, in which all the $k_f$ are unity, Equations (7) and (9) simplify drastically to the forms given in reference *1*, namely

$$m_i = c_i \sum_{j=1}^{i} u_j + (c_i - 1) \sum_{j=i+1}^{n} u_j; \; i = 1, \ldots, n-1 \qquad (11)$$

with

$$c_i = (n - i)/n \qquad (12)$$

Since the sum of products functions play an important role in the theory to be developed, several of their properties will be given here for future reference. The ranges of the indexes in the next four formulas are $t = 1, \ldots, n$; $t \leq u \leq v \leq w$, with the convention that

$$S(u+1, u) = S(v+1, v) \equiv 0 \qquad (13)$$

Most of the proofs involve splitting a sum of products function into two or three parts as follows:

$$S(t, w) = S(t, v) \prod_{h=v}^{w-1} k_h + S(v+1, w) \qquad (14)$$

$$= S(t, u) \prod_{h=u}^{w-1} k_h + S(u+1, v) \prod_{h=v}^{w-1} k_h + S(v+1, w) \qquad (15)$$

These two simple identities can be combined to derive a third used widely in the sequel:

$$S(t, u)\, S(u, w) - k_{u-1} S(t, u-1)\, S(u+1, w) = S(u, w)\, S(t, w) \qquad (16)$$

To prove this, use Equation (14) to expand $S(t, u)$ and $S(u, w)$, and use Equation (15) to expand $S(t, w)$.

Proof that Equations (8) and (9) give the performance desired is straightforward but tedious, and the details are omitted. Proof would involve substituting Equations (7), (8), and (9) into (3) to eliminate $m_{i-1}$ and $m_i$. The final result would be

$$y_i = \int_0^t \sum_{g=1}^{n} \prod_{h=g}^{n-1} k_h u_g \, dt / S(1, n); \; i = 1, \ldots, n \qquad (17)$$

Notice that the denominator contains a sum of products involving the disturbances $u_g$ in place of the capacity restrictions $y_p$. Since the right member of Equation (17) is the same for all levels $y_i$, it follows that

$$y_i = y_j; \; i, j = 1, \ldots, n; \; 0 \leq t \leq T \qquad (18)$$

Hence the levels move up and down together for this control law, no one level reaching its limit before any of the others.

This control scheme, whose instrument diagram is given in Figure 2, is called *feedforward* because the manipulations $m_i$ depend only on measurements of the disturbances $u_i$. Since feedforward control does not use measurements of the variables $y_i$ to be controlled, it has no way to compensate for unavoidable errors in the levels once they have occurred. Moreover, measurement errors and controller noise are accumulated rather than corrected. Although the feedforward control proposed in earlier articles is suitable for quick, rough response to upsets, it cannot make the fine readjustments on the levels themselves needed for good control.

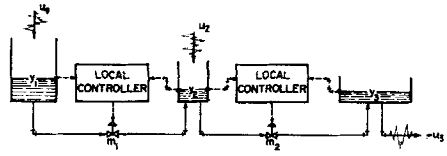


**Fig. 3. Flow and instrument diagram for local feedback control.**

## FEEDBACK

This article describes two feedback control schemes which make the manipulated variables $m_i$ depend directly on the controlled variables $y_i$ rather than on the diturbances $u_i$. Figure 3 is a block diagram for what will be called *local* feedback control. At this point the reader need only notice that as in all feedback schemes, the error signal to the controller comes from controlled variables instead of disturbances. The control law for the $n - 1$ controllers is

$$m_i = g_i\,(y_i - y_{i+1}); \; i = 1, \ldots, n-1 \qquad (19)$$

The input error signal $y_i - y_{i+1}$ is operated upon mathematically by the controller action $g_i(\;)$ to give the output signal to the manipulated variable $m_i$.

By suitable algebraic manipulations, an equation relating the $m_i$ to the disturbances $u_j$ can be obtained. In such an equation the manipulated variables at time $t$ would depend not on values of the $u_j$ at the same time $t$, but, because of the time lags in a feedback system, on earlier values. These delays stem from the time required for an upset to pass through the tanks. Consequently the manipulated variables will lag behind the values required by Equation (7), causing the levels to be somewhat out of balance temporarily. Even so, it is reasonable to require that the feedback system, given sufficient time after an upset, return to the condition described by Equation (18), all normalized tank levels equal.

To be precise, if disturbances take place for times between 0 and $t_1$, and if $t_2$ is some later time ($t_2 > t_1$), then the control scheme should be such that

$$\lim_{t_2 \to \infty} (y_i) = \lim_{t_2 \to \infty} (y_j), \; i, j = 1, \ldots, n \qquad (20)$$

A necessary and sufficient condition for this to happen is that the total amount of material transferred between any pair of tanks asymptotically match the amount called for by the ideal feedforward action. That is

$$\lim \int_0^{t_2} m_i \, dt = \sum_{j=1}^{n} a_{ij} \int_0^{t_1} u_j \, dt; \; i = 1, \ldots, n-1; \; 0 < t_1 < t_2 \qquad (21)$$

The equivalence of Equations (20) and (21) can be verified by integrating Equation (3) between 0 and $t_2$, substituting Equations (21), (9), and (8) as appropriate, passing to the limit, and noting that the two expressions can be equated to obtain Equation (20). Thus any feedback control law should have the asymptotic property described by Equation (21). The role of the constants $a_{ij}$ in Equation (21) led to naming them the *asymptotic* control constants.

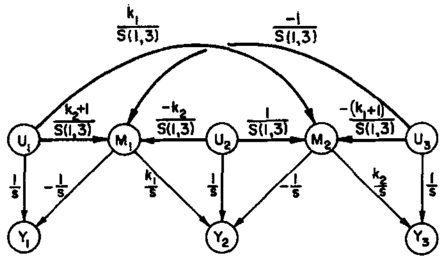


Fig. 4. Signal flow diagram for feedforward control.

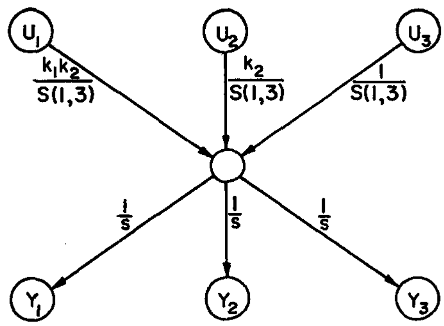


Fig. 5. Removal of nodes $M_1$ and $M_2$.

## SATURATION

Reference 1 dealt only with systems having unrestricted ranges on the manipulated variables $m_i$. In these ideal circumstances one needs no prior information about how these disturbances $u_i$ behave, for Equation (9) gives the proper control coefficients $a_{ij}$ in closed form for any set of upsets. Moreover, these coefficients are optimal not only at the terminal time $T$, but at all intermediate times as well.

Of more practical interest is the case where each manipulated variable $m_i$ must at all times lie between fixed, known limits $m_i'$ and $m_i''$:

$$m_i' \leq m_i \leq m_i''; i = 1, \ldots, n-1; 0 \leq t \leq \mathrm{T} \quad (22)$$

In these more realistic circumstances a feedforward scheme would usually not dare use the asymptotic control constants $a_{ij}$ because they could produce saturation, that is excessive fluctuations in the $m_i$. However, one can still employ a linear feedforward control law having the same form as Equation (7) but with different coefficients $b_{ij}$:

$$m_i = \sum_{j=1}^{n} b_{ij}\, u_j, i = 1, \ldots, n-1 \quad (23)$$

The $b_{ij}$, called the *feedforward control constants,* will usually be smaller in absolute value than the corresponding asymptotic constants in order to reduce the fluctuations in $m_i$. It will be shown that mild derivative action in the controllers will reduce saturation in a feedback system.

References 2 and 3 give techniques for finding suitable $b_{ij}$ for feedforward control. The constants obtained give nearly optimal probability of satisfactory operation for ideal feedforward control. One must know in advance the statistical character of the upsets to use these techniques. Although the constants are computed only for the terminal time $T$, in most cases they are also suitable at earlier times because the terminal situation is the most difficult to control when integrating elements such as tanks are present.

## LINEARITY

The system equations are linear, the only operations being integration with respect to time in the tanks and multiplication by constants for the flows. Moreover, only linear controllers (proportional-integral-derivative) will be considered, since such devices are in common use in the process industries. Hence it is convenient to apply the Laplace transformation to all functions of time and deal only with transfer functions of the complex frequency parameter $s$. For simplicity, all initial conditions are taken to be zero. Thus for example the Laplace transform $U_i(s)$ of the upset $u_i(t)$ is

$$U_i \equiv U_i(s) \equiv \int_0^\infty u_i(t)\, e^{-st}\, dt \quad (24)$$

By Equation (3) the transfer function for each tank is simply $1/s$, and the transfer function for the $i^{\text{th}}$ three-action controller can be written

$$G_i \equiv G_i(s) = K_i \left( D_i\, s + 1 + \frac{1}{I_i\, s} \right); \quad i = 1, \ldots, n-1 \quad (25)$$

The figures shown so far are material flow or instrumentation diagrams useful only for depicting the physical arrangement of equipment. In the analysis of information flow in a system it is more convenient to employ signal flow diagrams (4), in which transfer functions are represented by transmission arrows between signal nodes. Thus the flow and instrument diagram of Figure 2 is represented by the signal flow diagram of Figure 4.

Notice that although material always flows from tank $i$ to tank $i+1$, the signal flow is different because manipulated variable $M_i$ affects both tanks $i$ and $i+1$ on either side of it. In Figure 4 the light lines represent transmissions inherent in the uncontrolled physical system, while the heavy ones show the information channels added by the feedforward controllers. An equivalent signal flow diagram in which the intermediate signals $M_1$ and $M_2$ have been eliminated is shown in Figure 5. This diagram, which expresses Equation (17) graphically, confirms that the levels are at all times held in proportion to the tank capacities.

## LOCAL CONTROL

A minimum requirement for any feedback inventory control system is that its levels move up and down in constant ratio, if not immediately as in Equation (18), then at least asymptotically as in Equation (20). This suggests placing, between every pair of adjacent tanks, a controller manipulating the flow between tanks to hold the levels in constant ratio. To do this one constructs an error signal $E_i$ as follows:

$$E_i = Y_i - Y_{i+1}; i = 1, \ldots, n-1 \quad (26)$$

This signal is made the input to the $i^{\text{th}}$ controller so that

$$M_i = G_i\, E_i = G_i\, (Y_i - Y_{i+1}); i = 1 \ldots, n-1 \quad (27)$$

which is of course the transformed version of Equation (19). The error vanishes only when the levels $y_i$ and $y_{i+1}$ are equal, as given by Equation (18). Figure 6 is the

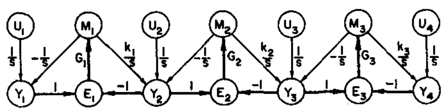


Fig. 6. Signal flow diagram for local control.

signal flow diagram for local control of a four-tank system, the instrument diagram for a three-tank system being Figure 3. The heavy lines show the transmissions introduced by the control system.

Although local control is easy enough to implement, its analysis is complicated by the controller interactions passing up and down the system through the tanks. Figure 7, in which the level and error signal nodes have been eliminated, shows how a disturbance is propagated throughout the system, ultimately affecting all the manipulated flows. Moreover, each flow fluctuation feeds back upon itself, not only negatively through the self-loops shown, but also positively through the other manipulated variables. Although positive feedback could lead to trouble, it turns out, fortunately, that even with this positive feedback the system is inherently stable for any three-action controller.

To prove this sweeping and not at all obvious statement with a minimum of effort, it is necessary to remove the self-loops of Figure 7 and rearrange things. Both the mathematical and signal flow algebras needed are straightforward (see reference 4, p. 18), and the desired results are

$$1 + \left[\frac{s}{G_i(k_i + 1)}\right] M_i = a_{i+1} U_i - a_{i+1} U_{i+1} + b_i M_{i-1} + a_{i+1} M_{i+1};\ i = 1, \ldots, n-1 \qquad (28)$$

where

$$a_i \equiv 1/(k_{i-1} + 1) = S(i-1, i-1)/S(i-1, i);\quad i = 2, \ldots, n-1 \qquad (29)$$

and

$$b_i \equiv k_{i-1}/(k_i + 1) = S(i+1, i+1)k_{i-1}/S(i, i+1);\quad i = 2, \ldots, n-1 \qquad (30)$$

This type of equation has already been studied in connection with unrefluxed multistage systems (absorbers for example), and the earlier results (5, 6, 7) can be adapted with little difficulty to the study of local control. Figure 8, the signal flow diagram corresponding to Equation (28), shows clearly the ladderlike structure analyzed in the earlier articles. The same structure arises in the study of the oscillations of masses situated at $n - 1$ points of a continuous elastic string or rod of finite length (8). Thus it is possible not only to apply many results already in the literature, but also to use mechanical and hydraulic analogies for visualizing the behavior of locally controlled inventory systems.

Although equations for the dynamic behavior of a local control system with three-action controllers can be derived, such expressions are too involved to be understood easily. Greater insight is gained by studying simpler situations demonstrating the effects of only a few parameters at a time. Having mastered the workings of these less complicated models, the engineer can, with little extra reflection, picture how a more sophisticated system would behave.

**Proportional Control**

Perhaps the simplest control action is the proportional band type in which controller output is proportional to

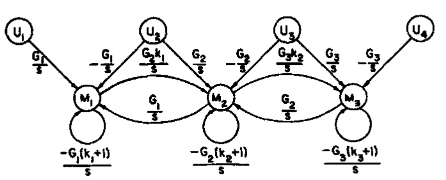

Fig. 7. Removal of error and level nodes.

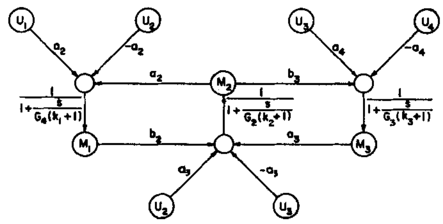

Fig. 8. Ladder structure of local control.

the input error signal. For proportional control, the controller transfer function is

$$G_i = K_i;\ i = 1, \ldots, n-1 \qquad (31)$$

and the dynamic elements of Equation (28) and Figure 8 reduce to those for a simple first-order system

$$\frac{1}{1 + \dfrac{s}{G_i(k_i + 1)}} = \frac{1}{1 + T_i s};\ i = 1, \ldots, n-1 \qquad (32)$$

where $T_i$ is the time constant given by

$$T_i = 1/K_i(k_i + 1);\ i = 1, \ldots, n-1 \qquad (33)$$

Since Equation (27) shows that the controller sensitivity $K_i$ has the dimensions (volume-time)$^{-1}$, the dimensions of the time constant must be time. Higher controller sensitivity brings a smaller time constant and hence faster response. When $n = 2$ and there is only one flow rate to manipulate, the sensitivity can be made as high as possible, since all first-order systems are stable. In fact, such a system will not even oscillate, no matter how high the sensitivity.

When many tanks are present, it is less easy to see what happens. Fortunately, a multitank system with local proportional control has exactly the same mathematical form as the typical multistage absorber system whose response to a feed concentration upset was analyzed in reference 6 and 7. Picture an $n - 1$ stage unrefluxed absorber with feed entering simultaneously at two adjacent stages. An upset in this feed concentration is propagated up and down the column, eventually affecting concentrations on every plate. Taken by itself each stage behaves as a first-order system if perfect mixing can be assumed, the time constant being proportional to the plate holdup. The response of the various plate concentrations to a feed disturbance has the same mathematical form as the response of the several manipulated flow rates to an upset in one of the tank levels. Hence several important results concerning local proportional control can be stated immediately without repeating the proofs.

Most important, such a system is stable for all values of controller sensitivity. Like a two-tank system, it will not even oscillate, no matter how high the sensitivities. Increased sensitivities, being analogous to diminished holdup in an absorber, always bring faster response. Moreover, the response acts to compensate for any upset rather than to aggravate it, never moving in the wrong direction.

The appendix proves that proportional control, given sufficient time after an upset, always balances out the levels as required by Equation (20). Hence proportional local control, simple as it is, has the required asymptotic properties for all positive values of the controller sensitivities.

**Offset To Steady Disturbances**

Suppose that a local proportional control system is upset by steady disturbances with constant magnitudes $\Upsilon_j$:

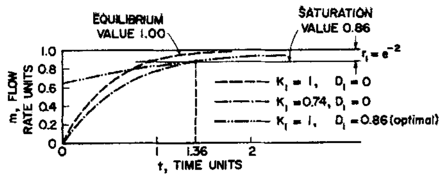


Fig. 9. Saturation in a two tank system $k_1 = 1, \gamma_1 = 1, \gamma_2 = -1$.

$$u_j(t) = \Upsilon_j;\ j = 1, \ldots, n \tag{34}$$

It is proven in the appendix that the manipulated variables $m_i(t)$ must satisfy the asymptotic relation (37), which upon transformation into the time domain by final value theorem (35) may be combined with Equation (34) and written as

$$\lim_{t_2 \to \infty} \int_0^{t_1} m_i(t)\, dt + \int_{t_1}^{t_2} m_i(t)\, dt = \sum_{j=1}^{n} a_{ij}\, \Upsilon_j \lim_{t_2 \to \infty} (t_2)$$

where $t_1 < t_2$. Both sides of this equation can be differentiated with respect to $t_2$ to obtain

$$\lim_{t_2 \to \infty} m_i(t_2) = \sum_{j=1}^{n} a_{ij}\, \Upsilon_j;\ i = 1, \ldots, n-1 \tag{35}$$

Let $e_i(\infty)$ represent the asymptotic value of the error $e_i$. Then by the above equation and Equations (27) and (31) defining proportional control

$$e_i(\infty) = \frac{1}{K_i} \lim_{t_2 \to \infty} m_i(t_2) = \frac{1}{K_i} \sum_{j=1}^{n} a_{ij}\, \Upsilon_j;\ i = 1, \ldots, n-1 \tag{36}$$

This offset error $e_i(\infty)$ does not vanish, which means that for steady disturbances the levels do not remain in the ratio required. There is therefore danger that if the steady disturbances persist long enough, some tank will fill or empty before the others. This disadvantage of local proportional control can be overcome by adding mild integral (reset) action.

#### Saturation Of A Manipulated Variable

Suppose that some set of steady disturbances $\Upsilon_j$ would require the asymptotic value $m_i(\infty)$ of some manipulated variable to fall outside the allowable range given by Equation (22). That is either

$$m_i(\infty) = \sum_{j=1}^{n} a_{ij}\, \Upsilon_j < m_i'$$

or

$$m_i(\infty) > m_i''$$

Then if these disturbances persist long enough, $m_i(t)$ will reach its limit and go no farther. When the manipulated variables represent production rates of processing units, this condition, known as *saturation* of $m_i$, may be undesirable because of the extra expense or danger of operating at extreme conditions.

Consider for example a two-tank conservative ($k_1 = 1$) system upset by unit steps, a positive one at tank 1 and a negative one at tank 2. The response of $m_i$ is given by

$$m_i = 1 - \exp(-2 K_1 t) \tag{37}$$

Figure 9 shows this required response for unit sensitivity ($K_1 = 1$) as a dashed line. If the upper limit on $m_1$ is 0.86 ($= 1 - e^{-2}$), then $m_1$ saturates when one unit of time has elapsed.

Since the disturbances studied here are in reality random variables, one might hope that any embarrassing set of upsets would change for the better before saturation occurs. Thus it would be worthwhile in any case to delay saturation as long as possible. This could be done for proportional control by decreasing the sensitivity as in Figure 9, in which the response for $K_1 = 0.86$ is shown to allow more time before saturation than does unit sensitivity. Unfortunately this remedy would also slow down the balancing of the levels, perhaps causing some level to reach a limit before the manipulated flows become great enough to carry off the excess material. A better solution to the saturation problem follows.

#### Proportional-Derivative Control

By combining derivative action with proportional control, one can delay saturation without sacrificing either sensitivity or desirable asymptotic properties. The controller transfer function for proportional-derivative control is

$$G_i(s) = K_i(D_i\, s + 1);\ i = 1, \ldots, n-1 \tag{38}$$

It is easily verified that the proportional component generates the asymptotic properties required by Equation (20).

To see how proportional-derivative action can delay saturation, consider a two-tank system with step loads of magnitudes $\Upsilon_1$ and $\Upsilon_2$. Combining Equations (28) and (38) and then inverting the transform one gets

$$m_1(t) = \left(\frac{\Upsilon_1 - \Upsilon_2}{k_1 + 1}\right) \left\{ 1 - \frac{1}{1 + K_1\,(k_1 + 1) D_1} \exp\left[\frac{-K_1(k_1 + 1)t}{1 + K_i(k_1 + 1) D_1}\right] \right\} \tag{38a}$$

Let $t_s$ be the time required for $m_1$ to reach its saturation value $m_1''$, and let $t_s^*$ be the maximum value that $t_s$ can attain for given sensitivity $K_1$ but adjustable derivative time $D_1$. Differential calculus gives the optimal derivative time $D_1^*$ as

$$D_1^* = (1 - r_1\, e)/r_1\, e(k_1 + 1) \tag{39}$$

where $e$ is the base of natural logarithms and $r_1$ is a dimensionless number defined by

$$r_1 = 1\,[m_1''(k_1 + 1)/(\Upsilon_1 - \Upsilon_2)] \tag{40}$$

In Figure 9 $r_1$ is shown to be $e^{-2} = 0.14$. Notice that derivative action is useful only if

$$r_1 < e^{-1} = 0.38 \tag{41}$$

that is if the saturation limit is not more than 38% below the equilibrium value required by the disturbance. When Equation (38*a*) is satisfied, Equation (39) may be used, and the maximum time until saturation is

$$t_s^* = [r_1\, e\, K_1\,(k_1 + 1)]^{-1} \tag{42}$$

In the hypothetical conservative system of Figure 9, the optimal derivative time for unit sensitivity is $(e - 1)/2 = 0.86$. The corresponding time until saturation is $e/2 = 1.36$ time units. In this case derivative action extends the time until saturation by a third. This is accomplished by immediate increase of $m_1$ to 62% of the equilibrium value. This anticipation not only reduces the need for later adjustment of $m_1$, but also brings the tank levels close to their desired values soon after the disturbance.

When there are more than two tanks, the interactions between the controllers complicate the equations for the $m_i$, obscuring the effects of the various derivative time adjustments. No simple method short of analogue simula-

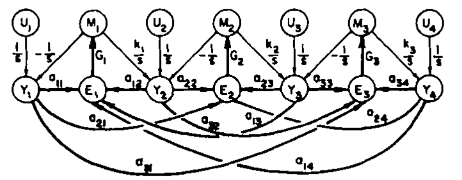

Fig. 10. Signal flow diagram for central control.

tion or direct solution of the equation is presently available for tuning the proportional-derivative controllers. Notice that excessively fast derivative action can make things worse instead of better. If in the example $D_i > (e^2 - 1)/2 = 3.2$, then $m_i$ will saturate immediately.

#### Three-Action Control

Integral action may be added to a proportional-derivative controller to make the offset error vanish. The result is a three-action or proportional-integral-derivative controller whose transfer function is given in Equation (25). Excessive integral action could produce oscillation, and the damping factor $\zeta$ is

$$\zeta_i = [I_i K_i (k_i + 1)/1 + D_i K_i (k_i + 1)]^{1/2}/2; \quad i = 1, \ldots, n-1 \qquad (43)$$

Thus to avoid oscillation, $\zeta_i$ must exceed unity, and

$$I_i > 4\,[D_i + K_i^{-1}(k_i + 1)^{-1}]^2; \; i = 1, \ldots, n-1 \qquad (44)$$

For a given sensitivity, derivative action increases the lower bound on the integral time, decreasing the rapidity with which offset error can be overcome.

### CENTRAL CONTROL

A fundamentally different feedback scheme for controlling a storage system is to make each error signal $e_i$ depend on all levels simultaneously. In this central control arrangement the levels are weighted by the asymptotic control constants $a_{ij}$ of Equation (7):

$$E_i = \sum_{j=1}^{n} a_{ij}\, \Upsilon_j; \; i = 1, \ldots, n-1 \qquad (45)$$

Figure 10 shows the signal flow diagram for a four-tank central control system. The heavy lines show the transmissions introduced by the control system.

Central control uses the same number of controllers $(n - 1)$ and level measuring devices $(n)$ as local control, but it requires considerably more feedback channels and coefficient multipliers $(n^2 - n)$ than local control, which has only $2(n - 1)$ channels. Despite this increased physical complexity, the analysis of central control is a great deal simpler than for local control because the additional channels compensate each other in such a way as to cancel out the interaction between the flows $m_i$. Thus when the error and level nodes are eliminated, the signal flow diagram breaks into the $n - 1$ independent pieces shown in Figure 11. This makes the controllers easier to tune because they do not interact. Moreover, it can be shown that this absence of controller interaction causes a centrally controlled system to settle down faster than one under local control.

#### Noninteraction

To prove there is no interaction between adjacent controllers, consider the transmissions between typical flows $m_i$ and $m_{i+1}$ $(i = 1, \ldots, n-2)$. Figure 10 shows that from $m_i$ to $m_{i+1}$ the transfer function is

$$M_{i+1}/M_i = G_{i+1}(-a_{i+1,i} + k_i\, a_{i+1,i+1})/s; \; i = 1, \ldots, n-2$$

But the factor on the right vanishes by Equations (9), and so

$$M_{i+1}/M_i = 0; \; i = 1, \ldots, n-2 \qquad (46)$$

Similarly, the transfer function from $m_{i+1}$ to $m_i$ vanishes by Equations (9):

$$M_i/M_{i+1} = G_i(-a_{i,i+1} + k_{i+1}\, a_{i,i+2})/s = 0; i = 1, \ldots, n-2 \qquad (47)$$

Since the flows do not interact, they can be written directly in terms of the disturbances as follows:

$$M_i = \left[ \sum_{j=1}^{n} a_{ij} U_j - (a_{ii} - k_i\, a_{i,i+1}) M_i \right] G_i/s$$

By Equations (9) and (14) the coefficient of $M_i$ in the right member can be shown to be unity. Hence

$$M_i = \left[ \sum_{j=1}^{n} a_{ij} U_j \right] \Big/ (1 + s\, G_i^{-1}) \qquad (48)$$

as shown in Figure 11.

#### Three-Action Central Control

The denominator of Equation (48) which contains the Laplacian parameter $s$ is exactly the same as for a locally controlled system with only two tanks. Hence all of the two-tank results already given (as guides to the behavior of locally controlled systems) can be applied immediately to centrally controlled systems. Proportional action gives the desired asymptotic properties, integral action corrects offset error, and derivative action can delay saturation. Equation (39) for optimal derivative time and Equation (44) for critical integral time can be applied to central control if the quantity $\Upsilon_1 - \Upsilon_2$ is replaced by $\sum_{j=1}^{n} a_{ij}\Upsilon_j$, where $\Upsilon_j$ are the magnitudes of step changes in the disturbances $U_j$.

### APPLICATIONS TO AUTOMATIC INVENTORY CONTROL

The situation studied is the prototype of an inventory control problem widespread in the petroleum industry. Despite the surprisingly large amount of capital tied up in stored materials, tanks, and their accouterments, there are at present no fully automatic inventory control systems. Instead, operators, engineers, and even plant managers spend valuable time manipulating production rates to hold tank levels within necessary limits. Alternatively, excessive storage is built to permit plants to operate at constant rates.

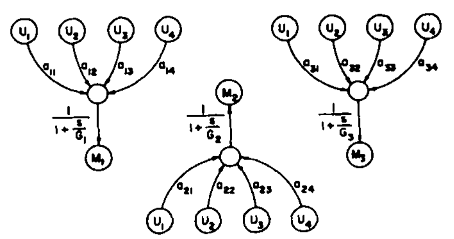

Fig. 11. Noninteracting structure of central control.

Although it is clearly impossible to hold all tank levels simultaneously constant, one can in principle maintain a constant ratio between each pair of levels, a policy known to give the maximum probability of keeping all levels within bounds. One way to do this is to base production rates on measurements of the shipments in and out of the system. Such a feedforward or open loop scheme, while easy to set up, suffers from the defect than any measurement errors or inaccuracies in the process description would accumulate and eventually force some level out of control.

Feedback or closed loop schemes, which base manipulated production rates on measurements of inventories instead of shipments, are free from these inadequacies, and two such plans, local and central control, have been presented. In local control each controller tries, by adjusting a single production rate, to match feed tank and product tank levels. The fact that a feed tank for one production unit can be a product tank for another causes the controllers to interact, causing disturbances to propagate throughout the system. Despite this, such ratio control action has been shown to be inherently stable if proportional or proportional-integral control is used. Moreover, proportional-integral control asymptotically brings the levels to a constant ratio as required for maximum probability of satisfactory operation. One can sometimes stave off short run saturation, running a production rate against one of its limits, by careful use of derivative action. Proportional local control would appear to be a reasonable way to administer a decentralized inventory control system in which the controllers are operators or managers. Each man would then be responsible only for the tanks adjacent to his plant.

In central control, the other feedback scheme analyzed, each controller input is a weighted average of all the levels in the system. Although this requires more communication channels than local control, the weighting coefficients happen to cancel out all controller interactions. From a dynamic standpoint this splits the system up into independent subsystems, one for each plant, simplifying the analysis and speeding up all responses. In a human system, a single person could examine all the levels and simultaneously generate the proper adjustments for all production rates. A central control system would have the desired inherent stability and asymptotic level matching of a locally controlled system, even though the former system reacts faster than the latter.

Tank gauging and product blending, once largely manual operations, are becoming increasingly automatic. The theory developed in this article suggests how to close the loop between the level signal and the production rates to give high probability of satisfactory operation with no danger of instability or offset error.

## ACKNOWLEDGMENT

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## NOTATION

$a_i$ = output transmission, Equations (28), (29)
$a_{ij}$ = asymptotic control constant, Equations (7), (9)
$b_i$ = input transmission, Equations (28), (30)
$b_{ij}$ = feedforward control constant, Equation (23)
$c_i$ = conservative control constant, Equations (11), (12)
$c_{ij}$ = constant, Equation (A7)
$d_i$ = approximant, Equations (A7), (A8)
$D_i$ = derivative time, Equation (25)
$e_i$ = approximant, Equations (A7), (A9)
$e_i(\infty)$ = offset error, Equation (36)
$E_i$ = transform of error, Equation (26)
$g_i(\ )$ = controller action, Equation (19)
$G_i$ = controller transfer function, Equation (25)
$i$ = tank number
$I_i$ = integral time, Equation (25)
$k_i$ = generation coefficient, Equation (1)
$K_i$ = controller sensitivity, Equation (25)
$m_i$ = instantaneous production deviation, Equation (3)
$m_i'$ = flow rate lower limit, Equation (22)
$m_i''$ = flow rate upper limit, Equation (22)
$\overline{m}_i$ = average flow rate (production schedule), Equation (1)
$n$ = number of tanks
$r_1$ = dimensionless number, Equation (40)
$s$ = Laplace parameter, Equation (24)
$S(u,v)$ = sum of products from $u$ to $v$ (sum of products function), Equation (8)
$t, t', t_1, t_2, T$ = time, Equations (3), (4), (20), (21)
$T_i$ = time constant, Equations (32), (33)
$u_i$ = instantaneous uncontrolled flow rate, Equation (7)
$\overline{u}_i$ = forecasted uncontrolled flow, Equation (1)
$x(t)$ = function of time, Equation (42)
$X^0$ = limiting value of transform, Equation (A1)
$y_i$ = instantaneous inventory deviation, Equation (3)
$y_i'$ = inventory deviation lower limit, Equation (6)
$y_i''$ = inventory deviation upper limit, Equation (6)
$\underline{y}_i$ = original inventory
$\zeta_i$ = damping coefficient, Equation (43)
$\Upsilon_i$ = unit step function, Equation (34)

Superscript zeroes are explained in Equation (A1).
Dummy indexes (subscripts): $g, h, i, j, t, u, v, w$.
Other capital letters denote Laplace transforms.

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## APPENDIX

It will be shown now that proportional control, given sufficient time after an upset, always balances out the levels as required by Equation (20). First it is convenient to express the equivalent condition (21) in terms of transforms. Let $X^0$ represent the limiting value of the Laplace transform $X(s)$ as $s$ approaches zero:

$$X^0 \equiv \lim_{s \to 0} X(s) \tag{A1}$$

The final value theorem states (9) that

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) \tag{A2}$$

Notice also that if $u_j(t)$ vanishes for $t > t_1$, then

$$\int_0^{t_1} u_j(t)dt = \lim_{t_2\to\infty}\int_0^{t_2} u_j(t)dt;\ j = 1, \ldots, n \quad \text{(A3)}$$

By substituting Equation (A3) into Equation (21), transforming, and then passing to the limit using the final value theorem, one obtains the condition desired:

$$M_i^0 = \sum_{j=1}^{n} a_{ij}U_j^0;\ i = 1, \ldots, n-1 \quad \text{(A4)}$$

When $s$ approaches zero, Equation (28) becomes

$$-b_iM^0_{i-1} + M_i^0 - a_{i+1}M^0_{i+1} = a_{i+1}U_i^0 - a_{i+1}U^0_{i+1};\ i = 1, \ldots, n-1 \quad \text{(A5)}$$

The solution to these equations is given in (6) as

$$M_i^0 = \sum_{j=1}^{n} (c_{ij}a_{j+1} - c_{i,j-1}a_j)\, U_j^0;\ i = 1, \ldots, n-1 \quad \text{(A6)}$$

where the constants $c_{ij}$ are given by

$$(d_i + e_i - 1)\, c_{ij} = \begin{cases} \prod_{h=i+1}^{j} a_h/e_h & \text{if } i < j \quad \text{(A7a)} \\ 1 & \text{if } i = j \quad \text{(A7b)} \\ \prod_{h=j+1}^{n-1} b_h/d_{h-1} & \text{if } i > j \quad \text{(A7c)} \end{cases}$$

The constants $d_i$ and $e_i$ are approximants (6, 10) to continued fractions formed from the products $a_h\, b_h$. They are defined recursively by

$$d_1 = 1 \quad \text{(A8a)}$$

$$d_h = 1 - a_h\, b_h/d_{h-1},\ h = 2, \ldots, n-1 \quad \text{(A8b)}$$

$$e_{n-1} = 1 \quad \text{(A9a)}$$

$$e_h = 1 - a_{h+1}b_{h+1}/e_{h+1}\ h = n-2, n-3, \ldots, n-1 \quad \text{(A9b)}$$

An equivalent condition, easier to prove, is obtained by summing on the second index from $j + 1$ to $n$, giving

$$\sum_{p=j+1}^{n} a_{ip} = -c_{ij}/(k_i + 1);\ i, j = 1, \ldots, n-1 \quad \text{(A10)}$$

Equation (A10) can be proven with the help of the following three expressions for the approximants in terms of sum of products functions:

$$d_h = S(h,h)\, S(h,n)/S(h,h+1)\, S(1,h);$$

$$h = 1, \ldots, n-1 \quad \text{(A11)}$$

$$e_h = S(h+1,h+1)\, S(h,n)/S(h,h+1)\, S(h+1,n);$$

$$h = 1, \ldots, n-1 \quad \text{(A12)}$$

$$d_h + e_h - 1 = S(h,h)\, S(h+1,h+1)\, S(1,n)/S(1,h)\, S(h,h+1)\, S(h+1,n);\ h = 1, \ldots, n-1 \quad \text{(A13)}$$

The proof of Equation (A11) is by induction on $h$, starting with $h = 1$, in which case Equation (A11) agrees trivially with definition (A8a). Next one writes $d_h$ in terms of sum of products functions by using Equations (A8b), (29), (30), and the induction hypothesis (A11). Placing the terms over a common denominator and using Equation (16) one gets Equation (A11), completing the induction. The strategy for proving Equation (A12) is similar, except that the induction goes in the opposite direction starting with $h = n - 1$, in which case the proof is immediate. For the induction step one needs Equation (16) to obtain $e_h$ from $e_{h+1}$. Derivation of Equation (A13) from Equations (A11) and (A12) requires the use of Equation (16) as shown below:

$$\begin{aligned}(d_h + e_h - 1)\, &S(1,h)\, S(h,h+1)\, S(h+1,n) \\ &= S(h,h)S(1,h+1)S(h+1,n) + S(1,h)[S(h+1,h+1)S(h,n) - S(h,h+1)S(h,n)] \\ &= S(h,h)[S(1,h+1)S(h+1,n) - k_h\, S(1,h)\, S(h+2,n)] \\ &= S(h,h)S(h+1,h+1)S(1,n)\end{aligned}$$

Equation (A10) and its equivalent Equations (A9) can now be proven. First consider the case where $j \geq i$. Then by Equations (A7), (A13), (8), and (9),

$$-c_{ij}/S(j,j+1) = -S(1,i)\, S(j+1,n)/S(1,n)$$

$$= -\sum_{p=j+1}^{n}\left[\prod_{h=p}^{n-1} k_h\, S(1,i)\right] \Big/ S(1,n) = \sum_{p=j+1}^{n} a_{ip} \quad \text{(A10)}$$

Now suppose $j < i$. Then application of Equations (9) and (8) to the left member of Equation (A10) gives

$$\sum_{p=j+1}^{n} a_{ip} = \sum_{p=j+1}^{i}\prod_{f=p}^{i-1} k_f\, S(i+1,n) - \sum_{p=j+1}^{n}\prod_{f=p}^{n-1} k_h\, S(1,i)$$

$$= S(i+1,n)[S(j+1,i) - S(1,i)];\ i = 2, \ldots, n;\ j < i \quad \text{(A14)}$$

The proof of Equation (A10) is by induction on $j = i, i - 1, \ldots, 1$. Equation (A10) can be proven now by a backward induction on $j$, starting with $j = i$ for which case Equation (A10) applies immediately. Suppose that the induction hypothesis (A10) holds for $j < i$. Then the chain of equations following shows that (A10) must also be true, which completes the induction. Application successively of Equation (8), (A7), (30) and (A11), the induction hypothesis Equation (A10), and Equation (A14) gives

$$\frac{-c_{ij}}{k_i + 1} = \frac{k_jS(1,j)[S(j+2,i)S(i+1,n) - S(i+1,n)S(1,i)]}{S(1,j+1)S(1,n)}$$

Equation (16) converts the right member above to the left member below, which is then reduced to the form desired by Equations (14) and (A14):

$$\frac{S(i+1,n)\,\{[S(1,j+1)S(j+1,i) - S(j+1,j+1)S(1,i)] - k_j\, S(1,j)S(1,i)\}}{S(1,j+1)S(1,n)}$$

$$= \frac{S(i+1,n)\{S(1,j+1)S(j+1,n) - S(1,i)\,[k_jS(1,j) + S(j+1,j+1,j+1)]\}}{S(1,j+1)S(1,n)}$$

$$= S(i+1,n)[S(j+1,i) - S(1,i)]/S(1,n)$$

$$= \sum_{p=j+1}^{n} a_{ip}$$

This completes the proof that proportional local control, simple as it is, has the required asymptotic properties for all positive values of the controller sensitivities.

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